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Realization of synchronization in time-delayed systems with stochastic perturbation

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Abstract
Since noise is ubiquitous in both nature and artificial systems, the stochastic perturbation influence on the dynamics of the unidirectionally coupled Ikeda models is investigated in this paper. On the one hand, sufficient conditions on the complete synchronization between these noise-perturbed and chaotic models are mathematically established, and an estimation of the sample transverse Lyapunov exponent is rigorously derived. On the other hand, specific examples and their numerical simulations are provided to illustrate the feasibility of our theoretical results. Moreover, the results on the Ikeda models are further generalized to a wide class of coupled nonlinear systems with multiple time delays and a common additive noise. It is believed that the idea and approach developed in this paper could be further generalized to investigate some other problems on chaos synchronization and chaos control with stochastic perturbation.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Various synchronization phenomena in two or more coupled oscillators modelled either by continuous systems or by discrete systems has received a great amount of attention since synchronization is of fundamental importance in many complex systems and permeating all kinds of sciences, ranging from physical to biological, from chemical to computer and even to social sciences [1–5]. In particular, chaos synchronization, due to its potentially practical application in secure communications, in modelling brain activity, and in pattern recognition phenomena, has attracted a growing interest over the last decade [6–12]. Several types of synchronization have been observed experimentally and defined theoretically in recent studies: completed synchronization, lag synchronization, phase synchronization and generalized synchronization. For a more accurate description of these types of synchronization, refer to [13–16], a recently comprehensive review [17] and references therein.

As is known to all, noise is omnipresent in both nature and man-made systems. Therefore, investigation of the noise effect on the synchronization phenomenon in the noise-perturbed chaotic system has become an important research topic. Actually, different noise might play a different (destructive or constructive) role in a different sense of synchronization. On the one hand, as for the destructive effect, it is of our common knowledge. The main concern of this kind of effect is how to accurately estimate the stochastic perturbation destructing the synchronization in the coupled continuous systems. A series of researches [18, 19] have been devoted to this topic, showing a clear relation between the noise intensities and coupling parameters to guarantee a successful complete or generalized synchronization in the coupled continuous systems with stochastic perturbation. On the other hand, the enhanced occurrence of complete or phase synchronization is of a constructive role when various noise coupling terms are taken into account in chaotic oscillators. The constructive effect of noise in these physical systems and chemical oscillators are quite similar to that in the famous phenomenon called stochastic resonance [20, 21]. As is reported in literature [22–25], the constructive role is always attributed to noise intrinsic property, for instance, nonzero mean of the added noise, or is due to noise influence on those orbits generated by the chaotic systems, for example, the residence time in the weak unstable region of the systems' orbits is probably reduced by the weakly additive noise however that in the region where the eigenvalues of the corresponding Jacobian matrix possess negative real part significantly augments. No matter which kind of noise effect is taken into account, the transverse Lyapunov exponent, describing the convergence rate of the orbits along the synchronization manifold, should be numerically or theoretically negative for an achievement of synchronization in the coupled systems with stochastic perturbation.

Moreover, recent papers [26–29] have consecutively come forth to investigate the relations between parameter mismatches and several types of synchronization in unidirectionally coupled chaotic systems with time delays. By utilizing the so-called Krasovskii–Lyapunov functional approach and numerically calculating the largest Lyapunov exponent transversal to the complete synchronization manifold, the authors of these papers, respectively, presented theoretical conditions and experimental techniques for realization of synchronization in many linearly or nonlinearly coupled systems, such as the Ikeda models and the Mackey–Glass systems. However, as is mentioned above, synchronization of concrete models is always subject to internal and external noise. For example, the coupling rate could be perturbed by some white noise; the coefficient in the driven system could be perturbed by the other white noise independent of the previous one; the whole systems might be settled in a rapidly fluctuating environment. Naturally, a question arises: 'Are there any theoretical sufficient conditions on the coupling parameters that guarantee a successful synchronization between two coupled and time-delayed chaotic models with stochastic perturbation?' The essential aim of the paper is to provide a positive answer of this question. In fact, we discuss the influence of some type of stochastic perturbation in the coupled chaotic systems with time delays. More concretely, we not only derive several conditions on complete synchronization between the coupled Ikeda models with some specific noise perturbation by virtue of the theoretical results on the stability of stochastic differential equations [33–37], but also generalize the approach to a wide class of nonlinear chaotic systems with multiple time delays and a common additive noise.

The rest of the paper is organized as follows. In section 2, preliminary Ikeda models with stochastic perturbation are basically depicted. In section 3, sufficient conditions on complete synchronization are rigorously presented; meanwhile, an accurate estimation of the sample transverse Lyapunov exponent is also theoretically given. Furthermore, some concrete models with specific parameters are provided to illustrate the feasibility of the theoretical results obtained in section 4, and a generalization of the approach to a wide class of nonlinear chaotic oscillators with multiple time delays and a common additive noise is performed in section 5. In section 6, the paper is closed with some conclusions and remarks. Besides, for self-containing and avoiding complicated notions, necessary criteria are introduced in the form of appendix.

2. Model description: the coupled Ikeda models with stochastic perturbation

As is known to all, the intrinsical dimension of differential equations with time delay could approach infinity if their time delay increasingly arrives at some critical value. This implies that continuous systems with time delay, even though their real dimension is less than three, still exhibit a variety of dynamics, including chaotic motion [30]. The Ikeda model, introduced to describe the dynamics of an optical bistable resonator, is such a prototype [31, 32]. This model could be mathematically given in the form of

$$\mathrm{d}x(t) = \{-\alpha x(t) + m\sin[x(t-\tau)]\}\,\mathrm{d}t, \tag{1}$$

where the state variable x represents the phase lag of the electric field across the resonator, α is the relaxation coefficient for the phase lag x, m denotes the laser intensity injected into the systems and τ stands for the round-trip times of the light in the resonators or feedback delayed times in the systems. Figure 1 numerically shows chaotic attractors generated by the Ikeda model (1) with specific parameters in different phase planes. The orbits and trajectories of the Ikeda model (1) in these figures exhibit complicated dynamics and are interweaving even in the two-dimensional phase plane, which is consistence with the infinite-dimensional property of differential equations with time delay.

In this paper, we adopt model (1) as a driving system, take the model governed by

$$\mathrm{d}y(t) = \{-\alpha y(t) + m\sin[y(t-\tau)] + [K + \vartheta_K \dot{w}_K(t)]\cdot z(t) + \vartheta_D \dot{w}_D(t)\cdot z(t-\tau)\}\,\mathrm{d}t$$

as a response system, where the state variable y stands for the response phase lag of the electric field across the resonator, K represents the coupling strength between the driver x and the response y, and coupling term $z(t)$ denotes the error between the x and y by $z(t) = x(t) - y(t)$. White noises $\dot{w}_K(t)$ and $\dot{w}_D(t)$ are either the same or mutually independent. Constants ϑ_K and ϑ_D are noise strengths. More realistically, the white noise with strength ϑ_K usually could be regarded as an environmental disturbance to the coupling strength K, and the white noise with strength ϑ_D is supposed to be induced by the fluctuation of the time delay.

In what follows, we write the above particular response system in a more general form

$$\mathrm{d}y(t) = \{-\alpha y(t) + m\sin[y(t-\tau)] + K\cdot z(t)\}\,\mathrm{d}t + \sigma(z(t), z(t-\tau), t)\,\mathrm{d}W(t), \tag{2}$$

where $W(t) = [w_1(t), \ldots, w_m(t)]^\top$ mathematically stands for an m-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathrm{F}, \mathbb{P})$ with a natural filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. Hence, each $\dot{w}_i(t) = \frac{\mathrm{d}w_i(t)}{\mathrm{d}t}$ becomes commonly white noise, and is assumed to be mutually independent of $\dot{w}_j(t)$ for every i, j with $i \neq j$. Besides, the term $\sigma : \mathbb{R}\times\mathbb{R}\times\{[-\tau, 0]\cup \mathbb{R}_+\} \to \mathbb{R}^{1\times m}$ is called the noise intensity row-matrix-valued function, and $\sigma(0, 0, t) \equiv 0$.

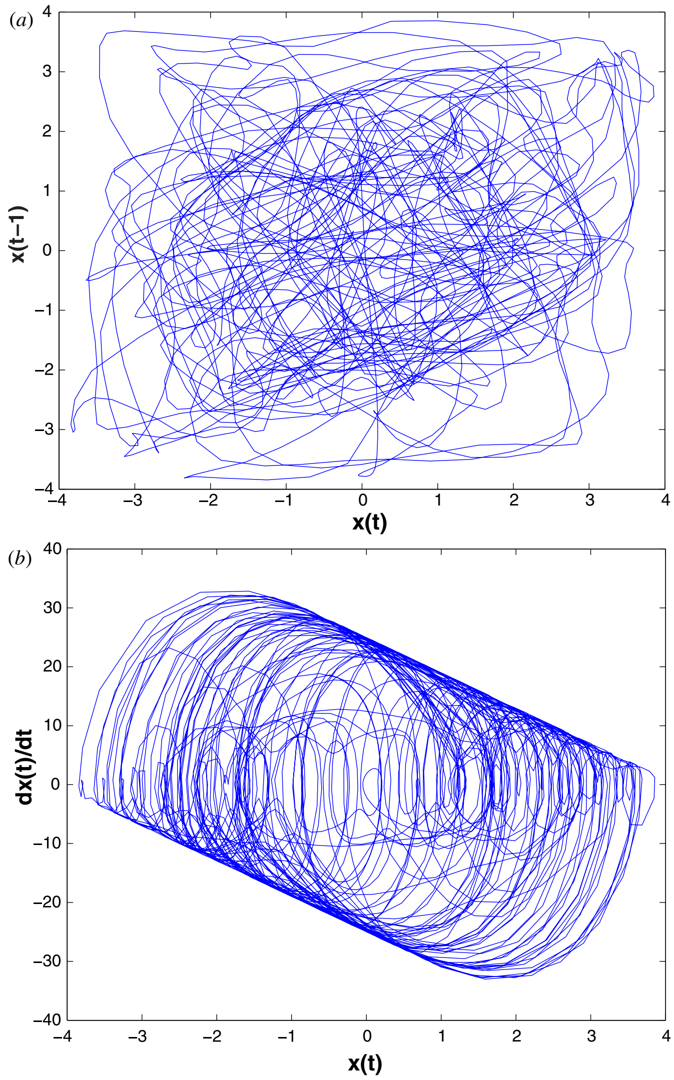

Figure 1. Chaotic attractors, generated by the Ikeda model (1), are, respectively, plotted in the $x(t)$ versus $x(t-1)$ plane (*a*) and in the $x(t)$ versus $\dot{x}(t)$ plane (*b*). All the corresponding parameters in model (1) are taken as $\alpha = 6, m = 25, \tau = 1$. The initial function $\phi_1(t) \equiv -2(-1 \leqslant t \leqslant 0)$ and the time-step size $\delta t = 0.01$.

Clearly, for the above particular response system, the intensity matrix-valued function is $\sigma(z, v) = [\vartheta_K z, \vartheta_D v]$ and $W(t) = [w_K(t), w_D(t)]^\top$ if these noises are mutually independent. It is worth mentioning that the noise term imported here is of multiplicative case. This is reasonable because this type of stochastic perturbation can be regarded as a result from the internal error and the bias of the real model. When particular oscillators are concretely established for simulations, inaccurate design or rapidly environmental fluctuation on the coupling strength and some other components in the systems cannot be avoidable. This thus leads us to import stochastic perturbation in the response system. Also note that the noise term here is of a multiplicative form interpreted in the sense of Itö which implies that the dependence of the process $x(t) - y(t)$ on the white noise $\dot{W}(t)$ at the same instant time t could be neglected. Although it seems to be a mathematical limiting procedure, this interpretation of multiplicative noise is reasonable when the rapidity of the environmental fluctuations is far less than the macroscopic time scale intrinsically possessed by concrete systems, such as the external fluctuations in a large class of biological and economic dynamical systems [48]. Above expatiation thus shows that the noise form imported in the response system is not much artificial but of a real significance. In addition, since there is an equivalent transformation between the noise term in the Stratonovich interpretation and in the Itö sense, the following discussion also could be adapted to the systems where the Stratonovich noise term is considered.

Consider the complete synchronization between systems (1) and (2) in the physical sense. Then from the driving system (1) and the response system (2), it yields their error dynamics

$$\mathrm{d}z(t) = \{-(\alpha + K)z(t) + m\cos[\xi(t)]z(t-\tau)\}\,\mathrm{d}t - \sigma(z(t), z(t-\tau), t)\,\mathrm{d}W(t), \tag{3}$$

where $\xi(t)$, in light of the classical mean value theorem, is a time-varying number in between the interval $(x(t-\tau)\wedge y(t-\tau), x(t-\tau)\vee y(t-\tau))$. Here, the notions $\wedge$ and $\vee$ stand for taking the minimum and maximum from two given numbers, respectively. Through out this paper, it is assumed that the term $\sigma(u, v, t)$ is locally Lipschitz continuous and not divergent as fast as the square of its components, thus satisfying the linear growth condition that will be more accurately illustrated in the following section. Hence, from the mathematical viewpoint, it follows from [35] that the error dynamics (3) possesses a unique global solution on $t \geqslant 0$, denoted by $z(t;\phi)$ for any initial data $\phi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R})$. This conclusion is consistent with the hypothesis (i) of criterion A in the appendix. Here, $C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R})$ represents the family of all $\mathcal{F}_0$-measurable bounded $C([-\tau, 0]; \mathbb{R})$-valued random variables, in which $C([-\tau, 0]; \mathbb{R})$ denotes the sets of all continuous functions from $[-\tau, 0]$ to $\mathbb{R}$.

It is not hard to see that $z(t; 0) \equiv 0$ is a trivial solution of the error dynamics (3). We say the trivial solution is globally asymptotically attractive in the physical sense if and only if $\lim_{t\to\infty} z(t;\phi) = 0$ for almost every initial datum $\phi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R})$. It should be mentioned that the trajectory generated by system (3), initiating from nonzero initial datum, will never approach but be convergent to the asymptotically stable trivial solution. This means that the influence of the above-described noise on the coupled systems will not disappear but asymptotically attenuate provided that the trivial solution could be controlled as a globally asymptotical attractor.

3. Complete synchronization between coupled Ikeda models with stochastic perturbation

3.1. Sufficient conditions on complete synchronization in a physical sense

As is mentioned in the preceding section, $z(t) = 0$ is a trivial solution of the error dynamics (3). That is, if this trivial solution is globally asymptotically attractive in the physical sense, the complete synchronization between systems (1) and (2) could be realized. Also it has been mentioned that the hypothesis (i) of criterion A in the appendix is manifestly contented for dynamics (3). So, in order to investigate the complete synchronization between systems (1) and (2) by this criterion, we only need to construct a function $V \in C^{2,1}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}_+)$ and determine the feasible region of coupling rate K and some other parameters, where the hypothesis (ii) of criterion A is also satisfied.

For this purpose, construct a function $V \in C^{2,1}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}_+)$ by $V(u, t) = u^2$. Thus, according to formula (A.2) in the appendix, the operation of $\mathcal{L}$ on the function V along with the error dynamics (3) gives

$$\mathcal{L}V(z(t), z(t-\tau), t) = 2z(t)\{-(\alpha+K)z(t) + m[\cos\xi(t)]\cdot z(t-\tau)\}$$
$$+\sigma(z(t), z(t-\tau), t)\sigma^{\top}(z(t), z(t-\tau), t). \quad (4)$$

In what follows, we are going to present an estimation of $\mathcal{L}V(z(t), z(t-\tau), t)$ in equation (4). Note that it has been already assumed in the last section that the noise intensity row matrix-valued function $\sigma(u, v, t)$ is controllable and not divergent as fast as the square of its components. Then, it is reasonable to presume that this matrix could be estimated by

$$\sigma(u, v, t)\sigma^{\top}(u, v, t) \leqslant \vartheta u^2 + \varsigma v^2, \quad (5)$$

where both ϑ and ς are non-negative constants. From the physical and engineering viewpoint, this assumption means that noise intensities due to inaccurate design and environmental fluctuation are at most linearly proportional to the amplitude of system states, so that the noise intensity could be controlled in some extent. In real application, we may observe and estimate the variation of noise intensities with systems states, and then, if necessary, design some equipment to suppress the noise influence to be somewhat consistent with the estimation (5). Naturally, both constants ϑ and ς become some controllable and adjustable parameters in the evaluation of this practical design.

Now, together with this estimation and equation (4), it yields the following estimation of $\mathcal{L}V(z(t), z(t-\tau), t)$:

$$\begin{aligned}\mathcal{L}V(z(t), z(t-\tau), t) &\leqslant -2(\alpha+K)z^2(t) + 2|m|\cdot|z(t)|\cdot|z(t-\tau)| + \vartheta z^2(t) + \varsigma z^2(t-\tau)\\ &\leqslant [-2(\alpha+K) + \vartheta + \varsigma + h]z^2(t) + 2|m|\cdot|z(t)|\cdot|z(t-\tau)|\\ &\quad + (\varsigma - h)z^2(t-\tau) - (\varsigma + h)z^2(t) + hz^2(t-\tau),\end{aligned}$$

where h is a selectively positive number and we will discuss the role of the number h acts in the complete synchronization between systems (1) and (2) later. On the other hand, if the following inequalities on the parameters hold:

$$\begin{aligned}&m^2 < (h-\varsigma)[2(\alpha+K) - \vartheta - \varsigma - h],\\ &\varsigma < h, \qquad -2(\alpha+K) + \vartheta + \varsigma + h < 0,\end{aligned} \quad (6)$$

then the symmetric matrix

$$\mathcal{A}(K, m) = \begin{bmatrix} -2(\alpha+K) + \vartheta + \varsigma + h & |m| \\ |m| & -h + \varsigma \end{bmatrix}$$

is definitely negative and its maximal eigenvalue, denoted by $-\rho$, consequently satisfies

$$0 > -\rho = -(\alpha+K) + \frac{\vartheta}{2} + \varsigma + \sqrt{m^2 + \left[-(\alpha+K) + \frac{\vartheta}{2} + h\right]^2} \geqslant -h.$$

Therefore, with the inequality condition (6), the estimation of $\mathcal{L}V(z(t), z(t-\tau), t)$ could be further given by

$$\begin{aligned}\mathcal{L}V(z(t), z(t-\tau), t) &\leqslant -(\rho+\varsigma+h)z^2(t) + (h-\rho)z^2(t-\tau),\\ &\triangleq -\omega_1(z^2(t)) + \omega_2(z^2(t-\tau)).\end{aligned} \quad (7)$$

Obviously, $\omega_1(u) > \omega_2(u)$ for any $u \neq 0$. Then, summarizing the above argument and utilizing criterion A, we obtain the following result on the sufficient conditions on complete synchronization between systems (1) and (2).

Conclusion 3.1. *The complete synchronization between the coupled Ikeda models (1) and (2) could be achieved for almost every initial data provided:*

(i) the intensity matrix is estimated by (5) for two non-negative constants ϑ and ς;
(ii) the coupling rate K and positive number h satisfy the inequality condition (6).

Note that once the term 'almost every' (a.e.) or 'almost surely' (a.s.) is adopted to describe a event, it means that this event could be realized with probability one. From the physical viewpoint, it implies that the complete synchronization between systems (1) and (2) could be surely realized provided that the conditions established above are satisfied.

Furthermore, the number h is an important adjustable parameter since different selection of it may lead to different constraint condition on the coupling strength K. For instance, if $h = \varsigma + 1$, we obtain

$$m^2 + \vartheta + 2\varsigma + 1 < 2(\alpha + K);$$

if $h = \alpha + K$, we have

$$m^2 < (\alpha + K)^2 - (\vartheta + 2\varsigma)(\alpha + K) + \varsigma(\vartheta + \varsigma). \tag{8}$$

Moreover, if $h = \alpha + K - \frac{\vartheta}{2}$, we get

$$m^2 < \left[(\alpha + K) - \frac{\vartheta}{2} - \varsigma\right]^2. \tag{9}$$

Therefore, the constraint of K is so much dependent on the choice of the positive number h, which implies that a proper selection of this number may lead to optimal estimations of the coupling strength K and other parameters. Furthermore, condition (9) shows that both the coupling strength K and the relaxation coefficient α positively contribute to the complete synchronization; nevertheless, both the laser intensity m and the noise intensity parameters ϑ and ς act an negative role. To be candid, this analysis accords with our common knowledge that the noise induced by the environmental fluctuation or the system bias usually plays a destructive role in chaos synchronization. However, we not only rigorously validate the fact but establish here a more accurate relation among these parameters and noise intensities for a successful complete synchronization as well. More importantly, our analytical results also guarantee a successful synchronization even when the noise influence is very strong.

3.2. Negative sample transverse Lyapunov exponent

Theoretical and numerical investigations of the so-called Lyapunov exponent are always performed so as to describe the various dynamical evolutions exhibited by specific systems. In particular, a negative transverse Lyapunov exponent corresponds to convergent dynamics with respect to the synchronization manifold, while a positive one indicates sensitive dependence on initial conditions and unsynchronized dynamics. In this subsection, we are to rigorously establish an estimation the sample transverse Lyapunov exponent of the systems (1) and (2), showing that the complete synchronization could be exponentially realized in the physical sense. For this purpose, we first assume that the following performed argument is based on the premises of conclusion 3.1. Thus, from the well-known Itô formula [33] and the estimation (7), it yields that for some positive λ,

$$\begin{aligned} e^{\lambda t} z^2(t) &= \phi^2(0) + \int_0^t \left\{ e^{\lambda s}[\lambda z^2(s) + \mathcal{L}z^2(s)] \right\} ds + \int_0^t 2\,e^{\lambda s} z(s)\sigma(z(s), z(s-\tau), s)\, dW(s) \\ &\leqslant \phi^2(0) + (\lambda - P)\int_0^t e^{\lambda s} z^2(s)\, ds + Q\int_0^t e^{\lambda s} z^2(s-\tau)\, ds \\ &\quad + \int_0^t 2\,e^{\lambda s} z(s)\sigma(z(s), z(s-\tau), s)\, dW(s) \\ &\triangleq \phi^2(0) + (\lambda - P)\int_0^t e^{\lambda s} z^2(s)\, ds + Q\int_0^t e^{\lambda s} z^2(s-\tau)\, ds + M(t), \end{aligned} \tag{10}$$

where the positive constants $P = \rho + \varsigma + h$, $Q = h - \rho$, and $P > Q$. Note a fact that

$$\int_{t-\tau}^{t} \mathrm{e}^{\lambda s} z^2(s)\,\mathrm{d}s = \int_{-\tau}^{t} \mathrm{e}^{\lambda s} z^2(s)\,\mathrm{d}s - \int_{0}^{t} \mathrm{e}^{\lambda(s-\tau)} z^2(s-\tau)\,\mathrm{d}s. \tag{11}$$

Multiplication of both sides of this equation by a factor $Q\,\mathrm{e}^{\lambda\tau}$ and adding this newly-obtained equation to estimation (10) produce

$$\begin{aligned} 0 &\leqslant \mathrm{e}^{\lambda t} z^2(t) + Q\,\mathrm{e}^{\lambda\tau} \int_{t-\tau}^{t} \mathrm{e}^{\lambda s} z^2(s)\,\mathrm{d}s \\ &\leqslant \phi^2(0) + Q\,\mathrm{e}^{\lambda\tau} \int_{-\tau}^{0} \mathrm{e}^{\lambda s} \phi^2(s)\,\mathrm{d}s + (\lambda - P + Q\,\mathrm{e}^{\lambda\tau}) \int_{0}^{t} \mathrm{e}^{\lambda s} z^2(s)\,\mathrm{d}s + M(t) \\ &\triangleq \widetilde{M}(t), \end{aligned}$$

which clearly implies that $\widetilde{M}(t)$ is non-negative. Also it is easy to verify that $M(t)$ is a real-valued continuous local martingale with $M(0) = 0$. Therefore, by utilizing criterion B in the appendix, we can conclude that $\lim_{t\to\infty} \widetilde{M}(t;\phi) < \infty$ a.s. provided that

$$\xi(\lambda) \triangleq \lambda - P + Q\,\mathrm{e}^{\lambda\tau} = 0. \tag{12}$$

Note that $P > Q$, $\xi(0) = -P + Q < 0$, and $\xi(P-Q) = -Q + Q\,\mathrm{e}^{(P-Q)\tau} > 0$. Thus, it is from the continuity of the function $\xi(\lambda)$ that equation (12) is solvable inside the interval $(0, P-Q)$. More accurately, there must exist at least a root $\lambda^* \in (0, P-Q) = (0, 2\rho + \varsigma)$ of equation (12), namely $\xi(\lambda^*) = 0$. Setting $\lambda = \lambda^*$, we obtain that

$$\limsup_{t\to\infty} \{\mathrm{e}^{\lambda^* t} z^2(t;\phi)\} < \infty, \qquad \text{a.s.} \tag{13}$$

Since conclusion 3.1 implies the trivial solution of the error dynamics (3) is almost surely globally asymptotically attractive, equation (13) consequently implies that the sample transverse Lyapunov exponent of the systems (1) and (2) could be expressed and estimated by

$$\limsup_{t\to\infty} \left\{ \frac{1}{t} \log[|z(t;\phi)|] \right\} \leqslant -\frac{\lambda^*}{2}, \qquad \text{a.s.} \tag{14}$$

Now, the above-performed argument could be concluded as the following result.

Conclusion 3.2. *Assume that all the hypotheses in conclusion 3.1 are satisfied. Then, not only the complete synchronization between the coupled Ikeda models (1) and (2) could be realized in the physical sense, but also the speed of approaching the complete synchronization manifold can be almost surely controlled by an exponential damping rate, that is, the upper estimation of the sample transverse Lyapunov exponent of the coupled Ikeda models can be restrained as (14).*

4. Illustrative examples

In this section, we provide some specific examples to show the feasibility of the above-established sufficient conditions on the complete synchronization between the coupled Ikeda models. Even though the choice of the parameters and the form of the stochastic perturbations designed below are somewhat synthetic, the possible application of the theoretical results is expressly illustrated.

Example 4.1. Take all the parameters and the noise intensity row matrix in systems (1)–(3) as

$\alpha = 6,\quad m = 25,\quad K = 34,\quad \tau = 1,\quad$ and $\quad \sigma(u, v, t) = -0.5u + 0.4v.$

Then, $\sigma(u, v, t)$ satisfies the locally Lipschitz continuous condition and the linear growth condition, i.e.,

$$\sigma^2(u, v, t) \leqslant 0.45u^2 + 0.36v^2.$$

Let $\vartheta = 0.45$ and $\varsigma = 0.36$. Therefore, it is not hard to verify that condition (8) is satisfied when $h = \alpha + K = 40$ and the corresponding maximal eigenvalue $-\rho$ of the matrix $\mathcal{A}(34, 25)$ is approximately equal to -14.4140. It is from conclusion 3.1 that the complete synchronization between systems (1) and (2) could be realized for almost every initial function $\phi \in C^b_{\mathcal{F}_0}([-1, 0]; \mathbb{R})$ of the error dynamics (3).

In addition, with the aid of Matlab program, we can numerically obtain an approximate solution of equation (11) with the above specific parameters, namely $\lambda^* \approx 0.7474$. Therefore, from conclusion 3.2, it follows that the sample transverse Lyapunov exponent of the coupled Ikeda models (1) and (2) is not larger than $-\lambda^*/2 \approx -0.3737$ for almost every initial function $\phi \in C^b_{\mathcal{F}_0}([-1, 0]; \mathbb{R})$ of the error dynamics (3). Accordingly, with these specific parameters, the speed of trajectory approaching the complete synchronization manifold can be almost surely controlled by an exponential damping rate which is larger than 0.3737.

By adopting the well-known Euler–Maruyama numerical scheme [38, 39], we, respectively, depict different sample paths of systems (1)–(3) in figures 2(*a*)–(*c*). Those sample paths show the evolution process of complete synchronization between the driving $x(t; \phi_1)$ and the response $y(t; \phi_2)$. It is noted that under some hypotheses the numerical solution given by Euler–Maruyama numerical scheme will converge to the true solution of the original system in an expectation sense as the sample time-step size δt tends to zero [40, 41]. However, these hypotheses on the continuity of the initial data, the local Lipschitz condition and square boundedness of the vector field are all satisfied for our examples and simulations. So, to some extent, all the sample paths plotted in figure 2 can reflect the true stochastic dynamical evolution of systems (1)–(3) if the sample time-step size is sufficiently small.

Example 4.2. Consider the parameters in systems (1)–(3) as $\alpha = 3, m = -18, \tau = 2$. Meanwhile, assume that $W(t) = [w_1(t), w_2(t)]^\top$ is a two-dimensional Brownian motion and the noise intensity matrix is in the form of

$$\sigma(u, v, t) = [A \sin 2u + B(\cos t)v, C \sin v],$$

where A, B, and C are undetermined parameters. Then,

$$\begin{aligned}\sigma(u, v, t)\sigma(u, v, t)^\top &= A^2 \sin^2 2u + 2AB(\cos t)(\sin 2u)v + (B\cos t)^2 v^2 + C^2 \sin^2 v \\ &\leqslant 4(A^2 + |AB|)u^2 + (B^2 + |AB| + C^2)v^2 \\ &\triangleq \vartheta u^2 + \varsigma v^2.\end{aligned}$$

Therefore, setting $h = \alpha + K - \frac{\vartheta}{2}$ and utilizing condition (9), we clearly have

$$\begin{aligned}K &> |m| + \frac{\vartheta}{2} + \varsigma - \alpha \\ &= 16 + 2A^2 + B^2 + C^2 + 2|AB|,\end{aligned} \tag{15}$$

which, due to conclusion 3.1, becomes a sufficient condition of the complete synchronization between systems (1) and (2) for almost every initial data.

Numerical simulations for a process of complete synchronization between systems (1) and (2) and unsynchronized dynamics are shown in figure 3(*a*). As a matter of fact, these unsynchronized dynamics are due to the selection of *K*, which violates condition (15). Numerical simulations for sample paths of the synchronized system (1) and (2) starting from different initial data are plotted in figure 3(*b*). A feasible region in the A–B–K space,

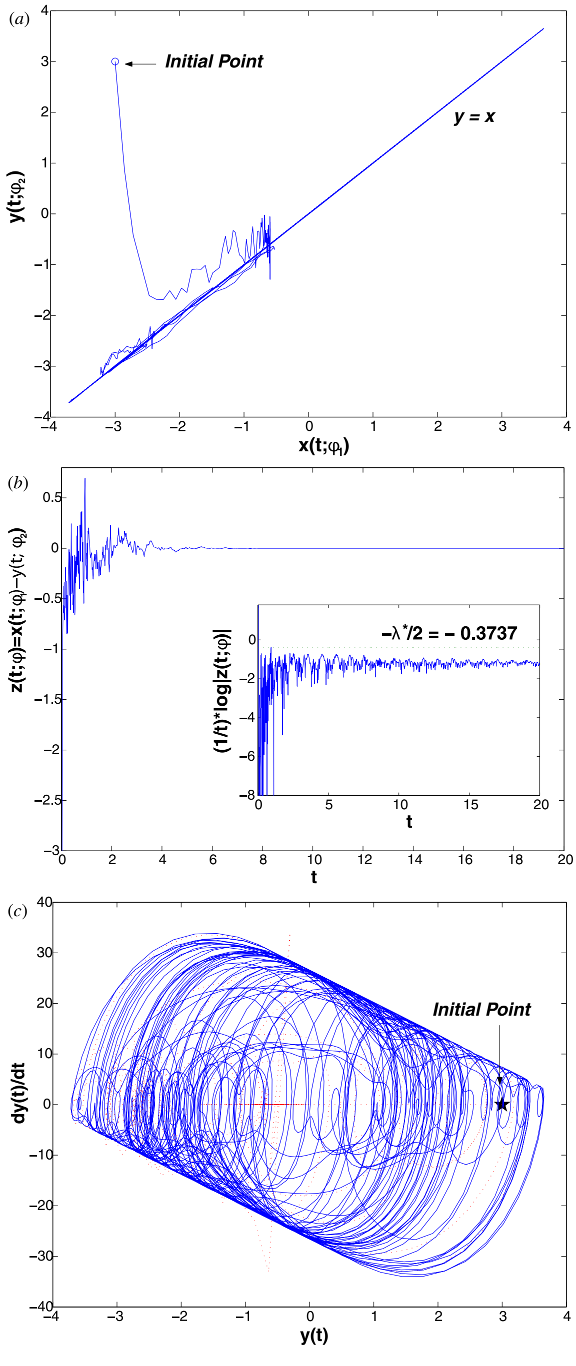

Figure 2. sample paths in the $x(t)$ versus $y(t)$ plane (a), in the t versus $z(t)$ plane (b) and in the $y(t)$ versus $\dot{y}(t)$ plane (c). The variation of $\frac{1}{t}\log|z(t;\phi)|$ with the time scale t is plotted at the bottom right corner of (b). The sample path from $t = 0$ to $t = 10$ is marked by dotted line in (c). All the corresponding parameters in systems (1) and (2) are taken as $\alpha = 6$, $m = 25$, $K = 34$ and $\tau = 1$. The initial functions $\phi(t) = \phi_1(t) - \phi_2(t) \equiv -3 - 3 = -6(-1 \leqslant t \leqslant 0)$ and the sample time-step size $\delta t = 0.01$. Initial point, highlighted by hollow dot or asterisk, denotes the position of the sample paths at $t = 0$.

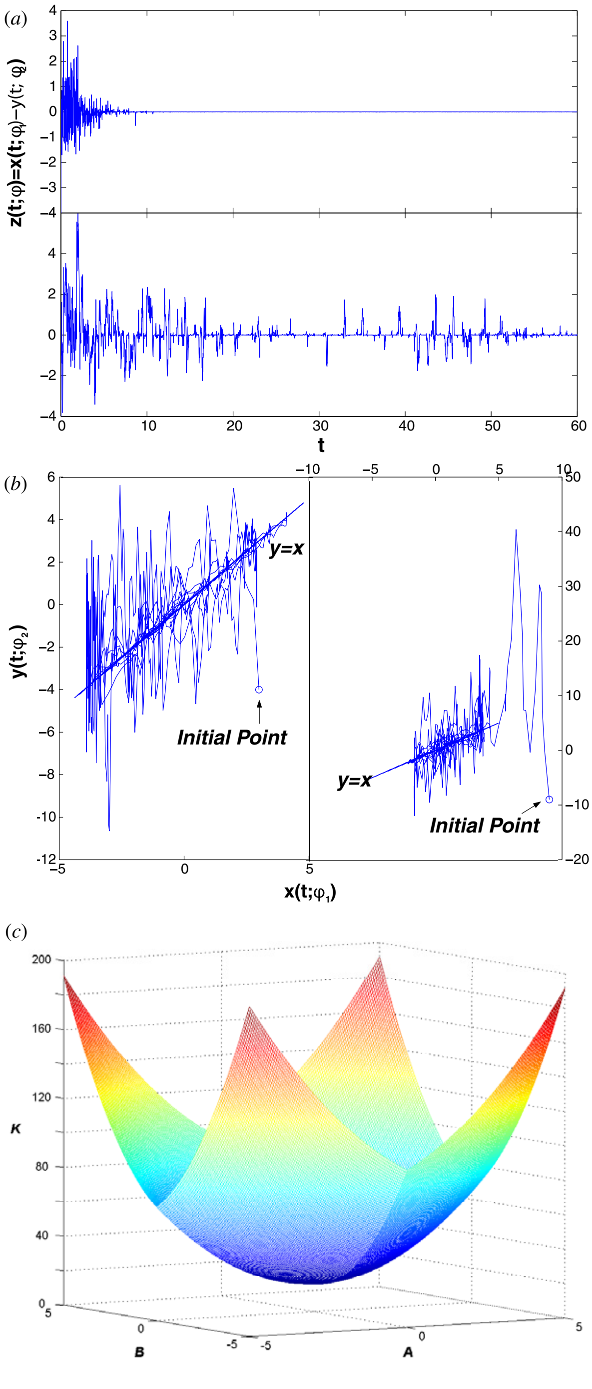

Figure 3. (*a*) Numerical simulations for synchronized ($K = 60$) and unsynchronized ($K = 9$) dynamics when $A = 3.5$, $B = -2$, $C = 1$ and initial functions $\phi_1(t) = 15$, $\phi_2(t) = 19$, $(-2 \leqslant t \leqslant 0)$. (*b*) sample paths, in the $x(t)$ versus $y(t)$ plane, starting from different initial functions, $\phi_1(t) = -\sin t + 3$, $\phi_2(t) = e^t - 5$ and $\phi_1(t) = -e^{2t} + 8$, $\phi_2(t) = \cos 2t - 10$, $(-2 \leqslant t \leqslant 0)$. Here, $A = -2$, $B = 0.5$, $C = 1.5$. (*c*) Feasible region is located above the plotted hyperplane in the A–B–K space when $|B| = |C|$. The sample time-step size $\delta t = 0.01$.

when $|B| = |C|$, is depicted in figure 3(*c*). The complete synchronization between (1) and (2) could be almost surely achieved provided that the values of all the parameters are chosen from this region. From this figure, we can see that the complete synchronization could be always realized even though the intensity of noise perturbation reaches a rather large value. We only need to design the coupling strength and other parameters to satisfy condition (15). This means that the above-described noise term is controllable in some extent.

5. Synchronization of generalized models

In this section, we are to generalize the above-performed approach to a wide class of coupled nonlinear systems with multiple time delays and a common additive noise. To this end, consider the complete synchronization between the driving system

$$\mathrm{d}x(t) = \{\mathcal{A}(x(t)) + F(x(t-\tau_1))\}\,\mathrm{d}t + \rho\,\mathrm{d}W^a(t), \tag{16}$$

and the response system

$$\begin{aligned}\mathrm{d}y(t) = {} & \{\mathcal{A}(y(t)) + F(y(t-\tau_1)) + G(z(t))\}\,\mathrm{d}t \\ & + \sigma(z(t), z(t-\tau_1), \ldots, z(t-\tau_\kappa), t)\,\mathrm{d}W(t) + \rho\mathrm{d}W^a(t),\end{aligned} \tag{17}$$

where $z(t)$ and $W(t)$ are the same meaning as those defined in section 2, and κ is the number of the time delays which might be different. The function $\mathcal{A}(u)$ satisfies

$$\frac{\mathcal{A}(u) - \mathcal{A}(v)}{u - v} \leqslant -\alpha,$$

for arbitrary $u \neq v$ and some constant α. The nonlinear function $F(u)$ satisfies

$$H_d \leqslant \frac{F(u) - F(v)}{u - v} \leqslant H_u,$$

for arbitrary $u \neq v$ and some constants H_u and H_d. Denote by H_1 the larger one between the constants $|H_u|$ and $|H_d|$. The coupling function $G(u)$ could be either linear or nonlinear, satisfying the locally Lipschitz condition and

$$G(0) = 0, \qquad H_2 \leqslant \frac{G(u) - G(v)}{u - v},$$

for arbitrary $u \neq v$ and some positive constant H_2. The noise intensity row matrix $\sigma(u, v_1, \ldots, v_\kappa, t)$ satisfies $\sigma(0, 0, \ldots, 0, t) = 0$ and

$$\sigma(u, v_1, \ldots, v_\kappa, t)\sigma^\top(u, v_1, \ldots, v_\kappa, t) \leqslant \vartheta u^2 + \sum_{i=1}^{\kappa} \varsigma_i v_i^2,$$

where ϑ and each v_i are non-negative constants. Apart from the above-illustrated terms, $\rho\mathrm{d}W^a(t)$ is a common additive noise and ρ is the noise strength. In particular, ρ is not so large that the additive noise may not destroy the original chaotic driving signal so much. Actually, the additive noise strength, though influencing the driving signal, will not at least obstruct the error dynamics of systems (16) and (17). Clearly, the error dynamics could be expressed by

$$\begin{aligned}\mathrm{d}z(t) = {} & \{\mathcal{A}(x(t)) - \mathcal{A}(y(t)) - G(z(t)) + F(x(t-\tau_1)) - F(y(t-\tau_1))\}\,\mathrm{d}t \\ & - \sigma(z(t), z(t-\tau_1), \ldots, z(t-\tau_\kappa), t)\,\mathrm{d}W(t),\end{aligned} \tag{18}$$

which is independent of the additive noise term. Similarly, construct a function $V \in C^{2,1}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}_+)$ by $V(u, t) = u^2$. Hence, analogous to formula (A.2) for system (A.1), a direct calculation along with the system (18) yields

$$\begin{aligned}\mathcal{L}V(z(t), z(t-\tau_1), \ldots, z(t-\tau_\kappa), t) = {} & 2[\mathcal{A}(x(t)) - \mathcal{A}(y(t))]z(t) - 2G(z(t))z(t) \\ & + 2[F(x(t-\tau_1)) - F(y(t-\tau_1))]z(t) \\ & + \sigma(z(t), z(t-\tau_1), \ldots, z(t-\tau_\kappa), t) \times \sigma^\top(z(t), z(t-\tau_1), \ldots, z(t-\tau_\kappa), t).\end{aligned}$$

Thus, from the above hypotheses on the functions $\mathcal{A}(u)$, $F(u)$, $G(u)$ and $\sigma(u, v_1, \ldots, v_\kappa, t)$, we can obtain an estimation

$$\begin{aligned}
&\mathcal{L}V(z(t), z(t-\tau_1), \ldots, z(t-\tau_\kappa), t) \\
&\qquad \leqslant -2(\alpha + H_2)z^2(t) + 2H_1 \cdot |z(t)| \cdot |z(t-\tau_1)| + \vartheta z^2(t) + \sum_{i=1}^{\kappa} \varsigma_i z^2(t-\tau_i) \\
&\qquad \leqslant -\left[\bar{\rho} + \sum_{i=1}^{\kappa}(\varsigma_i + h_i)\right] z^2(t) + \sum_{i=1}^{\kappa} (h_i - \bar{\rho})\, z^2(t-\tau_i),
\end{aligned}$$

where α, each H_i and h_i are the constants satisfying that the symmetric matrix

$$\widetilde{\mathcal{A}} = \begin{bmatrix} \vartheta - 2(\alpha + H_2) + \sum_{i=1}^{\kappa}(\varsigma_i + h_i) & H_1 & \mathbf{0} \\ H_1 & \varsigma_1 - h_1 & \mathbf{0} \\ \mathbf{0}^\top & \mathbf{0}^\top & \mathcal{D} \end{bmatrix}$$

is definitely negative and its maximal eigenvalue, denoted by $-\bar{\rho}$, is negative. Here, $\mathbf{0} \in \mathbb{R}^{1\times(\kappa-1)}$ is a zero row vector and $\mathcal{D} = \mathrm{diag}\{\varsigma_2 - h_2, \ldots, \varsigma_\kappa - h_\kappa\}$ is a diagonal matrix. In fact, the matrix $\widetilde{\mathcal{A}}$ is definitely negative provided that the following inequalities simultaneously hold:

$$\begin{aligned}
&H_1^2 < (h_1 - \varsigma_1)\left[2(\alpha + H_2) - \vartheta - \sum_{i=1}^{\kappa}(\varsigma_i + h_i)\right], \qquad -2(\alpha + H_2) + \vartheta + \sum_{i=1}^{\kappa}(\varsigma_i + h_i) < 0, \\
&\qquad \varsigma_i < h_i (i = 1, \ldots, \kappa).
\end{aligned} \tag{19}$$

Now, similar to the argument performed in section 3.2, we can obtain that for some positive μ,

$$\begin{aligned}
\mathrm{e}^{\mu t} z^2(t) &\leqslant \varphi^2(0) + \left[\mu - \bar{\rho} - \sum_{i=1}^{\kappa}(\varsigma_i + h_i)\right] \int_0^t \mathrm{e}^{\mu s} z^2(s)\, \mathrm{d}s \\
&\quad + \sum_{i=1}^{\kappa} (h_i - \bar{\rho}) \int_0^t \mathrm{e}^{\mu s} z^2(s-\tau_i)\, \mathrm{d}s + \widehat{M}(t),
\end{aligned} \tag{20}$$

where $\varphi \in C^b_{\mathcal{F}_0}([-\tau_M, 0]; \mathbb{R})$ is an initial datum of the error dynamics (18), $\tau_M = \max\{\tau_1, \ldots, \tau_\kappa\}$, and $\widehat{M}(t) = \int_0^t 2\,\mathrm{e}^{\lambda s} z(s)\sigma(z(s), z(s-\tau_1), \ldots z(s-\tau_\kappa), s)\, \mathrm{d}W(s)$ is a real-valued continuous local martingale with $\widehat{M}(0) = 0$. Together with the inequality (20) and the integral property in equation (11) for each time delay τ_i, we have

$$\begin{aligned}
0 &\leqslant \mathrm{e}^{\mu t} z^2(t) + \mathrm{e}^{\mu \tau_M} \sum_{i=1}^{\kappa}(h_i - \bar{\rho}) \int_{t-\tau_i}^t \mathrm{e}^{\mu s} z^2(s)\, \mathrm{d}s \\
&\leqslant \varphi^2(0) + \left[\mu - \bar{\rho} - \sum_{i=1}^{\kappa}(\varsigma_i + h_i) + \mathrm{e}^{\mu\tau_M} \sum_{i=1}^{\kappa} (h_i - \bar{\rho})\right] \int_0^t \mathrm{e}^{\mu s} z^2(s)\, \mathrm{d}s \\
&\quad + \sum_{i=1}^{\kappa}(h_i - \bar{\rho})\, \mathrm{e}^{\mu\tau_M} \int_{-\tau_i}^0 \mathrm{e}^{\mu s} \varphi^2(s)\, \mathrm{d}s + \widehat{M}(t) \\
&\triangleq \mathcal{M}(t).
\end{aligned}$$

Therefore, it follows from criterion B in the appendix that $\lim_{t\to\infty}\mathcal{M}(t;\phi) < \infty$ a.s. and

$$\limsup_{t\to\infty}\left\{\frac{1}{t}\log[|z(t;\varphi)|]\right\} \leqslant -\frac{\mu^*}{2}, \quad \text{a.s.}, \tag{21}$$

where μ^* is a positive root of the following equation with respect to the variable μ:

$$\mu - \bar{\rho} - \sum_{i=1}^{\kappa}(\varsigma_i + h_i) + \mathrm{e}^{\mu\tau_M}\sum_{i=1}^{\kappa}(h_i - \bar{\rho}) = 0.$$

Consequently, we approach the following conclusion.

Conclusion 5.1. *Assume that all the hypotheses in this section on the functions* $\mathcal{A}(u)$, $F(u)$, $G(u)$ *and* $\sigma(u, v_1, \ldots, v_\kappa, t)$ *are satisfied and inequalities (19) hold. Then, not only the complete synchronization between the coupled generalized models (16) and (17) could be realized in the physical sense, but also the upper boundary of the sample transverse Lyapunov exponent of the coupled models can be estimated by (21).*

As a matter of fact, conclusion 5.1 shows sufficient conditions on how to design the coupling function $G(u)$ for an achievement of the complete synchronization between the coupled systems (16) and (17) with stochastic perturbation. This implies that the synchronization might be realized in concrete simulations even though some of or all of the above hypotheses are violated. Besides, it could be found that the complete synchronization could be achieved in the physical sense even when the driving signal is destroyed by the common additive noise with a large strength. Obviously, the common additive noise does not play a role in our established results, actually influencing the dynamics of the chaotic driving signal. As reported in literature, this influence with a small amount of noise strength on the driving signal may profit the chaos synchronization. Thus, our theoretical result proposes a feasible and rigorous way to realize the complete synchronization when the strength of the common additive noise is relatively large.

6. Conclusion

In this paper, we have investigated the complete synchronization between the coupled Ikeda models with stochastic perturbation. By using the LaSalle-type criterion on the stability of stochastic differential equation with time delay and semi-martingale convergence theorem, we have shown that the complete synchronization could be almost surely realized and the corresponding sample transverse Lyapunov exponent is negative provided that some inequality conditions of coupling parameters and noise intensities are satisfied. Also, we have presented several specific examples and their simulation results, which illustrate the feasibility of the theoretical conditions established in this paper. In addition, we have generalized our approach to a wide class of coupled noised-perturbed chaotic systems with multiple time delays and a common additive noise, and given some extended results.

Since the world is replete with noise, either of destructive influence or of constructive effect, stochastic perturbation is unavoidable in real application of the synchronization theory. Then, a question on how the stochastic perturbation tampers the existing results for clean systems arises. Our results in the paper somewhat answer this question, showing that the complete synchronization could be almost surely achieved for a certain type of stochastic perturbation and coupling strength. Moreover, the idea and approach developed in this paper could be generalized to investigate different type of chaos synchronization in different chaotic oscillators with or without time delays, such as the Lorenz-like systems [42, 43], various circuit systems [44], the impulsive differential systems [45], the fractional differential equations [46] and various complex networks [47]. Therefore, we believe that our theoretical results established in the paper are of importance and might make some contribution to the related research field. As for the enhancement of chaos synchronization due to some types of noise in chaotic systems, the stochastic theory [33] allows us to make a further theoretical investigation of this dynamical scenario. The above-mentioned generalizations in various models and theoretical investigation on the constructive influence of noise, along with the development of theoretical and applicable criteria, become our current research topics [18, 19, 25].

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Appendix. Criteria for stability of stochastic differential equations with time delay

For the sake of self-containing of the paper, we introduce the criteria for the stability of stochastic differential equations with delay in this section. Consider the stochastic differential equation with time delay in the general form of

$$\mathrm{d}z(t) = f(z(t), z(t-\tau), t)\,\mathrm{d}t + \sigma(z(t), z(t-\tau), t)\,\mathrm{d}W(t). \tag{A.1}$$

$C^{2,1}(\mathbb{R}\times\mathbb{R}_+, \mathbb{R}_+)$ stands for the family of all non-negative functions $V(u,t)$ on $\mathbb{R}\times\mathbb{R}_+$ which are continuously twice differentiable in x and once differentiable in t. Then, for each $V \in C^{2,1}(\mathbb{R}\times\mathbb{R}_+, \mathbb{R}_+)$, the diffusion operator $\mathcal{L}$ along with system (A.1) could be expressed by

$$\mathcal{L}V(u,v,t) = V_t(u,t) + V_u(u,t)f(u,v,t) + \tfrac{1}{2}\sigma(u,v,t)\sigma^\top(u,v,t)V_{uu}(u,t), \tag{A.2}$$

where $V_t(u,t) = \frac{\partial V(u,t)}{\partial t}$, $V_u(u,t) = \frac{\partial V(u,t)}{\partial u}$ and $V_{uu}(u,t) = \frac{\partial^2 V(u,t)}{\partial u^2}$. Now, the LaSalle-type criterion for stochastic differential equation with delay could be summarized as follows.

Criterion A [34]. Assume that (i) system (A.1) possesses a unique solution on $t \geqslant 0$ for any given initial data belonging to $C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R})$, $f(u,v,t)$ and $g(u,v,t)$ are locally bounded in (u,v) and uniformly bounded in t. Assume also that (ii) there exist a function $V \in C^{2,1}(\mathbb{R}\times\mathbb{R}_+, \mathbb{R}_+)$, $\gamma \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ and $\omega_1, \omega_2 \in C(\mathbb{R}, \mathbb{R}_+)$ such that

$$\mathcal{L}V(u,v,t) \leqslant \gamma(t) - \omega_1(u) + \omega_2(v), \qquad (u,v,t) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+, \tag{A.3}$$

$$\omega_1(u) > \omega_2(u), \qquad \forall u \neq 0, \tag{A.4}$$

$$\lim_{|u|\to\infty} \inf_{0\leqslant t<\infty} V(u,t) = \infty. \tag{A.5}$$

Then it is valid that $\lim_{t\to\infty} z(t;\phi) = 0$ for almost every $\phi \in C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R})$.

The following criterion on semi-martingale convergence is used in the preceding section to investigate the sample transverse Lyapunov exponent of the driving system (1) and the response system (2).

Criterion B [36]. *Suppose that* $\widetilde{M}(t) \geqslant 0$ and it admits the following decomposition:

$$\widetilde{M}(t) = M_0 + A_1(t) - A_2(t) + M(t) \qquad \text{for} \quad t \geqslant 0,$$

where M_0 is a non-negative $\mathcal{F}_0$-measurable random variable with $\mathbb{E}[M_0] < \infty$, $A_1(t)$ and $A_2(t)$ are two continuous adapted increasing processes on $t \geqslant 0$ with $A_1(0) = A_2(0) = 0$ a.s., and $M(t)$ is a real-valued continuous local martingale with $M(0) = 0$ a.s.. Then,

$$\{\lim_{t\to\infty} A_1(t) < \infty\} \subset \{\lim_{t\to\infty} \widetilde{M}(t) < \infty\} \bigcap \{\lim_{t\to\infty} A_2(t) < \infty\} \text{ a.s.}$$

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